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Fundamental thermo-elasticity equations for thermally induced flexural vibration problems for inhomogeneous plates and thermo-elastic dynamical responses to a sinusoidally varying surface temperature

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Abstract Equations of motion governing thermally induced vibration of plates with inhomogeneous material properties through the thickness direction are presented. Equations of motion for thermally induced flexural vibration for inhomogeneous rectangular plates in which the material properties are given in the form of a power of the thickness coordinate are derived from the above-mentioned fundamental equations. An exact solution of the one-dimensional temperature change is presented for an inhomogeneous plate in which one surface is exposed to a sinusoidally varying temperature, and the other is kept at zero temperature change. The associated quasi-static and dynamic solutions pertinent to deflection and thermal stresses in the inhomogeneous rectangular plate are derived under the condition of simply supported edges. Numerical calculations are performed, and the effects of material inhomogeneity such as Young's modulus, coefficient of linear thermal expansion, and mass density, angular frequency in cyclic heating, and aspect ratio on the thermo-elastic response of the rectangular plate are shown in graphical form. Comparing the dynamic solutions with quasi-static ones, the effect of inertia on the thermo-elastic response of the inhomogeneous rectangular plate is evaluated.

Keywords Cyclic heating · Expansion theorem · Heat conduction · Natural mode · Rectangular plate

1 Introduction

The use of structural members under harsh conditions, such as high temperature and pressure, has increased in recent years, necessitating the development of various kinds of new materials. Functionally graded materials (FGMs) have received attention as one of these new materials, and are regarded for their potential applications in such fields as aerospace, engineering materials, opto-electronics, energy-conversion systems, and biomedical materials. The concept of FGMs [1] was proposed by Japanese researchers in the fields of aerospace and material science in 1984 for the purpose of developing advanced heat-resistant materials for a future space-plane. Since then FGMs have received attention, and a considerable amount of research on FGMs has been carried out all over the world. In FGMs, for the relaxation of thermal stress, ceramics are used on a surface exposed to surrounding high-temperature media so as to enable to obtain high heat resistance. Meanwhile metallic materials are used on the other surface cooled by a coolant so as to attain high thermal conductivity and high mechanical strength. Furthermore, a graded

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layer in which the material composition, microstructure and porosity are controlled beforehand to reduce thermal stresses caused by a mismatch of thermal expansion is introduced between the two surfaces. In the graded layer, the volume fraction of each material varies spatially. Therefore, the thermophysical and mechanical material constants vary with their composition. FGMs are treated as inhomogeneous materials in which the material constants vary spatially in theories on conduction of heat, elasticity and thermo-elasticity. Over the past two decades, various kinds of studies on FGMs have been conducted for problems in heat-conduction, elasticity and thermo-elasticity in which thermal and mechanical behaviors are analyzed for structural members, fracture, inverse problems and optimization.

When inhomogeneous materials are subjected to quasi-static mechanical or thermal loadings, the effect of inertia may be disregarded in most elastic and thermo-elastic problems; inertia terms are usually omitted from the equations of thermo-elasticity. In the models, wave propagation and vibration, which may arise in structural members, are disregarded. Solutions which are obtained under this simplification are often referred to as “static” or “quasi-static”. Vibration induced by heating may occur in thin-walled flexible structures. This is referred to as thermally induced vibration. Solutions which are obtained while considering inertia terms in the thermo-elasticity equations are often referred to as “dynamic”.

The effect of inertia on the thermo-elastic response has been considered in dynamical thermo-elastic problems for homogeneous materials. Danilovskaya [2,3] examined a dynamical thermo-elastic problem in a half-space consisting of a homogeneous elastic medium which is suddenly exposed to an ambient medium with a high temperature on the boundary surface. Danilovskaya’s investigations are described in the book by Boley and Weiner [4]. The importance of the inertia effect was emphasized in a class of thermo-elastic wave-propagation problems. Boley [5] analyzed a thermally induced flexural vibration problem involving a simply supported rectangular beam, one of whose sides is subjected to a sudden application of heat. A simple formula for approximating the ratio of the maximum deflection, obtained by a forced-vibration analysis, was presented, taking into account the effect of inertia on static deflection. As a result of this, the effect of inertia was found to be important for a slender beam. Boley and Barber [6] studied thermally induced flexural vibrations of rectangular plates and beams under some typical heat applications. A simple formula for rapidly estimating the ratio of the maximum deflection of the dynamic solution to the maximum deflection of the static solution was presented in terms of a dimensionless parameter. It was shown that the effect of inertia became important for rapidly applied heat inputs and for thin-walled plates. Subsequently, Boley [7] developed a simplified method to analyze thermally induced flexural vibrations of heated beams and plates. A simple general formula was derived for the ratio of the maximum dynamic deflection to the maximum static deflection. However, the formula was only approximate, but it was sufficiently accurate for engineering purposes, and reflected the essential physical characteristics of the phenomenon. The effects of damping and axial or in-plane loads were also considered and approximate formulas were derived.

Ignaczak and Nowacki [8] considered thermally excited flexural vibrations of a plate of moderate thickness subject to a harmonically varying heat flow in time on the boundary surfaces under some mechanical boundary conditions. Cukic [9] considered transversal vibrations of plates, produced by a harmonic heating, by means of the theory of coupled thermo-elasticity. Jadeja and Loo [10] used the Galerkin method to analyze thermally induced vibration of a rectangular plate with one edge fixed and the other edges simply supported, which was subjected to a sinusoidal heat input on one surface while the other was insulated thermally. The occurrence of thermally induced vibrations was shown in a flexible boom of a satellite subjected to variable heating [11] or a split-blanket solar array due to self-shadowing [12]. Because these thermally induced vibrations may degrade system performance, either by undesirable displacements or by unexpected accelerations, these problems are of practical importance.

Papers on the analysis of vibration problems regarding structural members for inhomogeneous elastic media are relatively scarce. Tomar et al. [13] studied free axisymmetric vibrations of an isotropic, elastic, non-homogeneous circular plate of linearly varying thickness on the basis of the classical theory of plates. The non-homogeneity of the material of the plate was assumed to arise due to the exponential variation of Young’s modulus and mass density in the radial direction whereas Poisson’s ratio was assumed to remain constant. The frequency parameters corresponding to the first two vibration modes were computed for different values of the non-homogeneity parameter and thickness variation constant (taper constant) and for clamped and simply supported edge conditions of the plate. Horgan and Chan [14] treated the vibration of inhomogeneous strings, rods and membranes with continuously varying properties, and provided specific examples for which closed-form exact solutions were obtained. An integral-equation-based method for obtaining lower bounds for the vibration frequencies was described, and the effectiveness of the method was demonstrated.

Loy et al. [15] studied the vibration of cylindrical shells made of a functionally graded material composed of stainless steel and nickel. The frequency characteristics, the influence of the constituent volume fractions and the effects of the configurations of the constituent materials on the frequencies were demonstrated. Oh et al. [16] studied the thermo-elastic modeling and vibrations of thin-walled rotating blades turbomachinery made of functionally graded ceramic–metal-based materials. The governing dynamic equations expressed in terms of one-dimensional displacement measures were formulated for pre-twisted and tapered thin-walled beams, rotating at a constant angular velocity and exposed to a steady temperature field of a prescribed gradient through the blade-wall thickness. The numerical results of the effects of volume fraction, temperature gradient and taper ratio on the vibration characteristics were investigated taking into account the temperature dependence of the material properties. Vel and Batra [17] presented a three-dimensional exact solution for free and forced vibrations of simply supported functionally graded rectangular plates. Exact natural frequencies, displacements and stresses were used to assess the accuracy of the classical plate theory, the first-order shear-deformation theory and a third-order shear-deformation theory for functionally graded plates. Forced vibrations of a plate with a sinusoidally spatial variation of the pressure applied on its top surface were investigated.

When inhomogeneous materials such as FGMs are applied to structural materials in aerospace engineering, the structures are necessarily required to be thin-walled. Weight-saving in structural members often results in a considerable decrease in thickness and stiffness. When thin-walled members are subject to repeated loads due to fluctuating temperatures, an analysis of thermally induced vibration is essential to avoid breakage failure due to resonance and fatigue. However, a study which treats a thermally induced vibration of thin-walled structural members made of inhomogeneous materials has not yet appeared in the literature.

This is why the present authors have devoted themselves to the theoretical treatment of heat conduction and thermally induced vibration problems of thin-walled structural members composed of a continuous linear medium with isotropic and inhomogeneous material properties through the thickness direction. In [18] an exact solution was derived for a one-dimensional inhomogeneous beam undergoing a sinusoidal temperature variation at one of its boundaries. The effects of frequency in cyclic heating and material inhomogeneity on the dynamic response of deflection and thermal stresses were evaluated numerically for a simply supported beam and for a beam with one edge fixed and the other free. Here we present an analytical study of thermally induced flexural vibration for an inhomogeneous rectangular plate in the case of cyclic heating. First, the equations of motion governing thermally induced vibration of plates with inhomogeneous material properties through the thickness direction are considered. Subsequently, equations of motion of thermally induced flexural vibration for inhomogeneous rectangular plates in which the material properties are given in the form of a power of the thickness coordinate are derived from the above-mentioned fundamental equations. We consider an inhomogeneous plate in which one surface is exposed to a sinusoidally varying temperature, and the other is kept at zero temperature change. We derive the associated quasi-static and dynamic solutions pertinent to deflection and thermal stresses in the inhomogeneous rectangular plate for simply supported edges. We perform a numerical calculation, and show the results of temperature change in an unsteady state and the associated quasi-static behavior and dynamic response of deflection and thermal stresses of the plate. Comparing the maximum dynamic thermo-elastic response with the maximum quasi-static thermo-elastic behavior, we discuss the influence of frequency in cyclic heating, material inhomogeneity and aspect ratio on the inertia effect in the thermo-elastic response of the rectangular plate.

2 Equations of motion for inhomogeneous plates

We consider a plate of uniform thickness h which is subject to a distributed transverse load $p(x, y, t)$ and is exposed to a temperature change $T(x, y, z', t)$ from the stress-free state. We assume that the plate is composed of isotropic linear elastic material whose thermo-elastic properties are temperature-independent, and that the plate has inhomogeneous material properties through the thickness direction. The origin of the coordinate in the thickness direction is generally selected at the middle plane of the cross-section of the plate for thermal bending of a homogeneous plate [19,20]. If the origin of the coordinate in the thickness direction is appropriately selected in the cross-section of the inhomogeneous plate, in which Young's modulus has an arbitrary inhomogeneity in the thickness direction, thermal bending in the inhomogeneous plate can be treated easily. Thus, the coordinate in the thickness direction z whose position of the origin is located at $z' = \eta$ from the top surface $z' = 0$ of the plate is defined as

$$z = z' - \eta. \tag{1}$$

The position η of the origin of the coordinate z is defined as

$$\int_0^h E(z')(z' - \eta)\mathrm{d}z' = 0. \tag{2}$$

Consequently, the position η is determined as

$$\eta = \frac{\int_0^h E(z')z' \,\mathrm{d}z'}{\int_0^h E(z')\mathrm{d}z'}. \tag{3}$$

If it is assumed that line elements which originally are perpendicular to a reference plane of the plate remain straight and normal to the deformed reference plane, the displacement components u', v' in the x- and y-directions at a point x, y, z in the plate are expressed as

$$u' = u - z\frac{\partial w}{\partial x}, \quad v' = v - z\frac{\partial w}{\partial y}, \tag{4}$$

in which $u = u(x, y)$, $v = v(x, y)$ are the displacement components in the x- and y-directions at the reference plane $z = 0$, and $w = w(x, y)$ is the deflection of the plate.

Assuming that the linear strain-displacement relations are valid, we may write the in-plane strain components as follows:

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u'}{\partial x} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_{yy} = \frac{\partial v'}{\partial y} = \frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2}, \\ \varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - z\frac{\partial^2 w}{\partial x \partial y}. \end{aligned} \tag{5}$$

The corresponding stress components are

$$\begin{aligned} \sigma_{xx} &= \frac{E}{1-\nu^2}\left\{\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} - z\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) - (1+\nu)\alpha T\right\}, \\ \sigma_{yy} &= \frac{E}{1-\nu^2}\left\{\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x} - z\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) - (1+\nu)\alpha T\right\}, \\ \sigma_{xy} &= \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y}\right), \end{aligned} \tag{6}$$

in which Young's modulus $E = E(z')$ and the coefficient of linear thermal expansion $\alpha = \alpha(z')$ vary with the thickness coordinate z', whereas Poisson's ratio ν is assumed to be constant.

Resultant forces per unit length N_x, N_y, N_{xy} and resultant moments per unit length M_x, M_y and M_{xy} with respect to the reference plane $z = 0$ are defined as

$$\begin{aligned} N_x &= \int_{-\eta}^{h-\eta} \sigma_{xx}\mathrm{d}z, \quad N_y = \int_{-\eta}^{h-\eta} \sigma_{yy}\mathrm{d}z, \quad N_{xy} = \int_{-\eta}^{h-\eta} \sigma_{xy}\mathrm{d}z, \\ M_x &= \int_{-\eta}^{h-\eta} \sigma_{xx}z\mathrm{d}z, \quad M_y = \int_{-\eta}^{h-\eta} \sigma_{yy}z\mathrm{d}z, \quad M_{xy} = -\int_{-\eta}^{h-\eta} \sigma_{xy}z\mathrm{d}z. \end{aligned} \tag{7}$$

In consideration of the relation (1), substitution of (6) in (7) gives

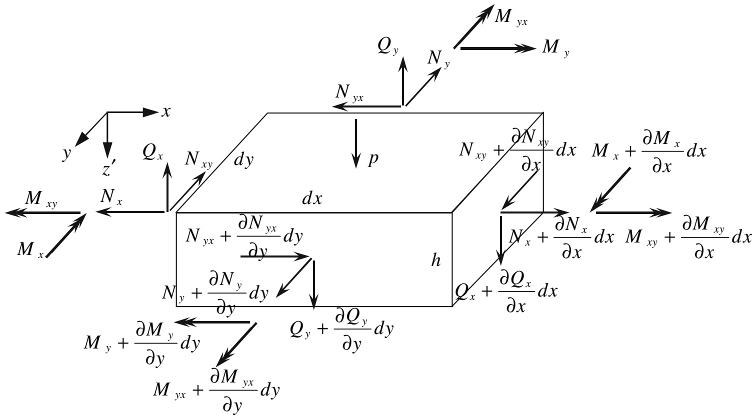

Fig. 1 Resultant forces and moments acting on plate element

$$N_x = c_1\left(\frac{\partial u}{\partial x}+\nu\frac{\partial v}{\partial y}\right)-\frac{1}{1-\nu}N_{\mathrm{T}},\quad N_y = c_1\left(\frac{\partial v}{\partial y}+\nu\frac{\partial u}{\partial x}\right)-\frac{1}{1-\nu}N_{\mathrm{T}},$$
$$N_{xy}=\frac{1}{2}(1-\nu)c_1\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \tag{8}$$
and
$$M_x = -c_3\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)-\frac{1}{1-\nu}M_{\mathrm{T}},\quad M_y = -c_3\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)-\frac{1}{1-\nu}M_{\mathrm{T}},$$
$$M_{xy}=(1-\nu)c_3\frac{\partial^2 w}{\partial x\partial y}, \tag{9}$$
where the coefficients c_1, c_3 and the thermal resultant force N_{T} and thermal resultant moment M_{T} are defined as
$$c_1=\frac{1}{1-\nu^2}\int_0^h E(z')\mathrm{d}z',\quad c_3=\frac{1}{1-\nu^2}\int_0^h E(z')(z'-\eta)^2\mathrm{d}z', \tag{10}$$
$$N_{\mathrm{T}}=\int_0^h \alpha(z')E(z')T(x,y,z',t)\mathrm{d}z',\quad M_{\mathrm{T}}=\int_0^h \alpha(z')E(z')T(x,y,z',t)(z'-\eta)\mathrm{d}z'. \tag{11}$$
Next, we consider the equations of motion for the inhomogeneous plate element shown in Fig. 1. Resultant shear forces per unit length Q_x, Q_y and resultant moment per unit length M_{yx} with respect to the reference plane $z=0$ are defined as
$$Q_x=\int_{-\eta}^{h-\eta}\sigma_{xz}\mathrm{d}z,\quad Q_y=\int_{-\eta}^{h-\eta}\sigma_{yz}\mathrm{d}z,\quad M_{yx}=\int_{-\eta}^{h-\eta}\sigma_{yz}z\mathrm{d}z. \tag{12}$$
The equation of motion for the z-direction is
$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+p=\mu\frac{\partial^2 w}{\partial t^2}\,, \tag{13}$$
in which μ is mass per unit area defined as
$$\mu=\int_0^h \rho(z')\mathrm{d}z', \tag{14}$$
where $\rho=\rho(z')$ denotes mass density.

Summation of moments about the y- and x-axes leads to the expressions
$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{yx}}{\partial y}-\mu_1\frac{\partial^2 u}{\partial t^2}+\mu_2\frac{\partial^3 w}{\partial x\partial t^2}=Q_x,$$
$$\frac{\partial M_y}{\partial y}-\frac{\partial M_{xy}}{\partial x}-\mu_1\frac{\partial^2 v}{\partial t^2}+\mu_2\frac{\partial^3 w}{\partial y\partial t^2}=Q_y, \tag{15}$$
in which

$$\mu_1 = \int_0^h \rho(z')(z'-\eta)\mathrm{d}z', \quad \mu_2 = \int_0^h \rho(z')(z'-\eta)^2\mathrm{d}z'. \tag{16}$$

Equations (9), (13) and (15) can be combined to obtain

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 w + \frac{\mu}{c_3}\frac{\partial^2 w}{\partial t^2} - \frac{\mu_2}{c_3}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) = \frac{1}{c_3}\left\{p - \frac{1}{1-\nu}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)M_{\mathrm{T}} - \mu_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}. \tag{17}$$

Next, the equations governing motion in the x- and y-directions are

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{yx}}{\partial y} = \mu\frac{\partial^2 u}{\partial t^2} - \mu_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial x}\right), \quad \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y} = \mu\frac{\partial^2 v}{\partial t^2} - \mu_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial y}\right). \tag{18}$$

If a fundamental natural frequency among longitudinal vibrations for the in-plane displacement components u, v at a reference plane is much larger than the natural frequency in flexural vibration for deflection, the inertia terms of the in-plane displacement components u, v may be ignored. Furthermore, ignoring rotary inertia, the dynamic response of deflection is governed by

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 w + \frac{\mu}{c_3}\frac{\partial^2 w}{\partial t^2} = \frac{1}{c_3}\left\{p - \frac{1}{1-\nu}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)M_{\mathrm{T}}\right\}. \tag{19}$$

Similarly, the equations governing the in-plane motions are, respectively,

$$\frac{\partial}{\partial x}\left(N_x + \mu_1\frac{\partial^2 w}{\partial t^2}\right) + \frac{\partial N_{yx}}{\partial y} = 0, \quad \frac{\partial}{\partial y}\left(N_y + \mu_1\frac{\partial^2 w}{\partial t^2}\right) + \frac{\partial N_{xy}}{\partial x} = 0. \tag{20}$$

Introducing a stress function χ defined as

$$N_x + \mu_1\frac{\partial^2 w}{\partial t^2} \equiv \frac{\partial^2 \chi}{\partial y^2}, \quad N_y + \mu_1\frac{\partial^2 w}{\partial t^2} \equiv \frac{\partial^2 \chi}{\partial x^2}, \quad N_{xy} \equiv -\frac{\partial^2 \chi}{\partial x \partial y}, \tag{21}$$

we observe that Eqs. (20) are satisfied identically.

Substitution of (21) in (8) and the equations of compatibility in the strain components yields the governing equation for the stress function χ as

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 \chi = (1-\nu)\mu_1\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) - \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)N_{\mathrm{T}}. \tag{22}$$

3 Analytical development

3.1 Heat conduction for inhomogeneous plates

We consider a plate of thickness h as shown in Fig. 2. The plate has inhomogeneous material properties which vary continuously in the thickness direction z'. The plate, which initially is at zero temperature, is bounded by the planes $z'=0$ and $z'=h$. The surface $z'=h$ is kept at zero temperature change while the surface $z'=0$ is exposed, for

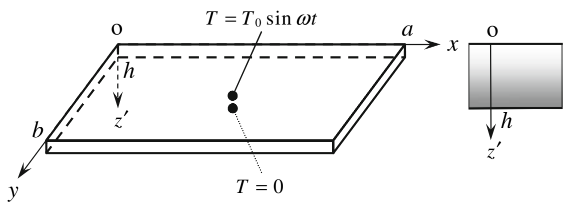

Fig. 2 Geometry and conditions

time $t > 0$, to a prescribed temperature which varies sinusoidally in time with amplitude T_0 and angular frequency ω. The fundamental equation of heat conduction in an unsteady state for the plate is given as

$$c(z')\rho(z')\frac{\partial T}{\partial t} = \frac{\partial}{\partial z'}\left\{\lambda(z')\frac{\partial T}{\partial z'}\right\}, \tag{23}$$

in which $c\rho$ is the specific heat capacity; λ is the thermal conductivity.

The corresponding initial condition and the thermal boundary conditions are expressed as

$$T = 0,\ t = 0, \tag{24}$$

$$T = T_0 \sin\omega t,\ z' = 0, \tag{25}$$

$$T = 0,\ z' = h. \tag{26}$$

It is assumed that the specific heat capacity $c\rho$ and the thermal conductivity λ are independently given in the form of a power of the thickness coordinate z':

$$c(z')\rho(z') = c_0\rho_0\left(1 + \frac{z'}{h}\right)^k, \quad \lambda(z') = \lambda_0\left(1 + \frac{z'}{h}\right)^l \tag{27}$$

Here the constants c_0, ρ_0, λ_0 are typical quantities of specific heat, mass density and thermal conductivity, respectively. The exponents k, l are parameters representing the inhomogeneities in the specific heat capacity and the thermal conductivity, respectively.

The following dimensionless quantities are defined as

$$\bar{T}(\zeta,\tau) = \frac{T(z',t)}{T_0}, \quad \zeta = 1 + \frac{z'}{h}, \quad \tau = \frac{\kappa_0}{h^2}t, \quad \bar{\omega} = \frac{h^2}{\kappa_0}\omega, \tag{28}$$

in which κ_0 is a typical value of the thermal diffusivity.

$$\kappa_0 = \frac{\lambda_0}{c_0\rho_0}. \tag{29}$$

Solving the fundamental equation (23) under the conditions (24)–(26) with the dimensionless variables defined in (28) by means of the Laplace transform and the residue theorem, we may write the solution for the temperature change, in dimensionless form, in terms of Bessel functions with real order and complex arguments as

$$\begin{aligned}\bar{T}(\zeta,\tau) = &-\sum_{j=1}^{\infty} \mathrm{e}^{-q_j^2\tau}\frac{2q_j\bar{\omega}}{q_j^4+\bar{\omega}^2}\frac{\zeta^{p_1}}{F_1'(q_j)}\left\{J_\xi(p_3q_j\zeta^{p_2})Y_\xi(2^{p_2}p_3q_j) - Y_\xi(p_3q_j\zeta^{p_2})J_\xi(2^{p_2}p_3q_j)\right\}\\ &+\frac{1}{2\mathrm{i}}(\cos\bar{\omega}\tau + \mathrm{i}\sin\bar{\omega}\tau)\frac{\zeta^{p_1}}{F_1(q_j)}\left\{J_\xi(p_3z_1\zeta^{p_2})Y_\xi(2^{p_2}p_3z_1) - Y_\xi(p_3z_1\zeta^{p_2})J_\xi(2^{p_2}p_3z_1)\right\}\\ &-\frac{1}{2\mathrm{i}}(\cos\bar{\omega}\tau - \mathrm{i}\sin\bar{\omega}\tau)\frac{\zeta^{p_1}}{F_1(q_j)}\left\{J_\xi(p_3z_2\zeta^{p_2})Y_\xi(2^{p_2}p_3z_2) - Y_\xi(p_3z_2\zeta^{p_2})J_\xi(2^{p_2}p_3z_2)\right\},\end{aligned} \tag{30}$$

in which

$$\xi = \left|\frac{l-1}{k-l+2}\right| \tag{31}$$

$$p_1 = \frac{1}{2}(1-l), \quad p_2 = \frac{1}{2}(k-l+2), \quad p_3 = \frac{2}{|k-l+2|}, \tag{32}$$

$$z_1 = \left(\frac{\bar{\omega}}{2}\right)^{\frac{1}{2}}(1-\mathrm{i}), \quad z_2 = \left(\frac{\bar{\omega}}{2}\right)^{\frac{1}{2}}(1+\mathrm{i}), \quad i = \sqrt{-1}, \tag{33}$$

$$F_1(q) = J_\xi(p_3q)Y_\xi(2^{p_2}p_3q) - Y_\xi(p_3q)J_\xi(2^{p_2}p_3q), \tag{34}$$

$$F_1'(q) = -p_3\left[\left\{J_{\xi+1}(p_3q)Y_\xi(2^{p_2}p_3q) - Y_{\xi+1}(p_3q)J_\xi(2^{p_2}p_3q)\right\}\right. \\ \left. + 2^{p_2}\left\{J_\xi(p_3q)Y_{\xi+1}(2^{p_2}p_3q) - Y_\xi(p_3q)J_{\xi+1}(2^{p_2}p_3q)\right\}\right]. \quad (35)$$

The characteristic value q_j is the jth positive root which satisfies the following transcendental equation

$$J_\xi(p_3q)Y_\xi(2^{p_2}p_3q) - Y_\xi(p_3q)J_\xi(2^{p_2}p_3q) = 0. \quad (36)$$

Now we transform the solution of the temperature change (30) into expression in terms of real elementary functions. Supposing a relation between the exponents k and l as

$$k = -l \quad (-1 < k, l < 1), \quad (37)$$

then Eq. (31) becomes

$$\xi = \frac{1}{2}. \quad (38)$$

The following relations between Bessel functions with half-odd-numbered order and elementary functions hold:

$$J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\sin z, \quad J_{3/2}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left(\frac{1}{z}\sin z - \cos z\right), \\ Y_{1/2}(z) = -\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cos z, \quad Y_{3/2}(z) = -\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left(\frac{1}{z}\cos z + \sin z\right). \quad (39)$$

Transforming (30) with the relations (39), we may write the solution of temperature change $\bar{T}$ as

$$\bar{T}(\zeta, \tau) = \sum_{j=1}^{\infty} D_{1j}(\zeta)\mathrm{e}^{-q_j^2\tau} + D_2(\zeta)\cos\bar{\omega}\tau + D_3(\zeta)\sin\bar{\omega}\tau, \quad (40)$$

in which

$$D_{1j}(\zeta) = -\frac{2\bar{\omega}q_j}{q_j^4 + \bar{\omega}^2}\frac{(-1)^j}{p_3(1-2^{p_2})}\sin\left(p_3(\zeta^{p_2} - 2^{p_2})q_j\right), \\ D_2(\zeta) = \frac{1}{\sin^2\beta\cosh^2\beta + \cos^2\beta\sinh^2\beta}(\sin\alpha\cosh\alpha\cos\beta\sinh\beta - \cos\alpha\sinh\alpha\sin\beta\cosh\beta), \\ D_3(\zeta) = \frac{1}{\sin^2\beta\cosh^2\beta + \cos^2\beta\sinh^2\beta}(\sin\alpha\cosh\alpha\sin\beta\cosh\beta + \cos\alpha\sinh\alpha\cos\beta\sinh\beta) \\ \alpha = p_3\left(\frac{\bar{\omega}}{2}\right)^{\frac{1}{2}}(\zeta^{p_2} - 2^{p_2}), \quad \beta = p_3\left(\frac{\bar{\omega}}{2}\right)^{\frac{1}{2}}(1 - 2^{p_2}). \quad (41)$$

Substitution of (38) in (32) gives

$$p_2 = 1 - l, \quad p_3 = \frac{1}{|1-l|}. \quad (42)$$

Substituting Eqs. (38), (39) in (36), we may express the characteristic value q_j as

$$q_j = \frac{j\pi}{p_3(2^{p_2} - 1)} \quad j = 1, 2, \ldots. \quad (43)$$

3.2 Quasi-static thermo-elastic behavior of inhomogeneous plates

We assume that Young's modulus E and the coefficient of linear thermal expansion α are independently given in the form of a power of the thickness coordinate z' as

$$E(z') = E_0\left(1 + \frac{z'}{h}\right)^m, \quad \alpha(z') = \alpha_0\left(1 + \frac{z'}{h}\right)^n, \quad (44)$$

in which the constants E_0, α_0 are typical quantities representing Young's modulus and the coefficient of linear thermal expansion, and the exponents m, n are parameters representing the inhomogeneities in Young's modulus and the coefficient of linear thermal expansion, respectively.

The following dimensionless quantities are defined:

$$(\bar{x}, \bar{y}) = \frac{(x, y)}{a}, \quad \bar{b} = \frac{b}{a}, \quad \bar{h} = \frac{h}{a}, \quad \bar{E}(\zeta) = \frac{E(z')}{E_0}, \quad \bar{\alpha}(\zeta) = \frac{\alpha(z')}{\alpha_0}, \quad \bar{\eta} = \frac{\eta}{h}, \quad \bar{c}_3 = \frac{c_3}{E_0 h^3}, \\ \bar{w} = \frac{w}{\alpha_0 T_0 h}, \quad \bar{M}_T = \frac{M_T}{\alpha_0 E_0 T_0 h^2}, \quad (\bar{\sigma}_{xx}, \bar{\sigma}_{yy}, \bar{\sigma}_{xy}) = \frac{(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})}{\alpha_0 E_0 T_0}, \tag{45}$$

in which a, b are lengths of the rectangular plate.

With the dimensionless quantities in (45), Eqs. (44) lead to

$$\bar{E}(\zeta) = \zeta^m, \quad \bar{\alpha}(\zeta) = \zeta^n. \tag{46}$$

The position of the reference plane η and flexural rigidity c_3 are given in dimensionless form as

$$\bar{\eta} = \begin{cases} \dfrac{(m+1)(2^{m+2}-1)}{(m+2)(2^{m+1}-1)} - 1, & m \neq -2, -1, \\ 2 \log 2 - 1, & m = -2, \\ \dfrac{1}{\log 2} - 1, & m = -1, \end{cases} \tag{47}$$

$$\bar{c}_3 = \begin{cases} \dfrac{1}{1-\nu^2} \left\{ \dfrac{2^{m+3}-1}{m+3} - \dfrac{(m+1)(2^{m+2}-1)^2}{(m+2)^2(2^{m+1}-1)} \right\}, & m \neq -3, -2, -1, \\ \dfrac{1}{1-\nu^2} \left(\log 2 - \dfrac{2}{3} \right), & m = -3, \\ \dfrac{1}{1-\nu^2} \left\{ 1 - 2 (\log 2)^2 \right\}, & m = -2, \\ \dfrac{1}{1-\nu^2} \left(\dfrac{3}{2} - \dfrac{1}{\log 2} \right), & m = -1. \end{cases} \tag{48}$$

Disregarding the inertia term in (19), and considering that the plate is not subject to a transverse load, we may write the fundamental equation governing the quasi-static deflection in dimensionless form as

$$\left(\frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2} \right)^2 \bar{w} = -\frac{1}{1-\nu} \frac{1}{\bar{h}^2} \frac{1}{\bar{c}_3} \left(\frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2} \right) \bar{M}_T. \tag{49}$$

Supposing the plate is simply supported on a contour, mechanical boundary conditions for the deflection w can be given in dimensionless form as

$$\bar{w} = 0, \quad \frac{\partial^2 \bar{w}}{\partial \bar{x}^2} = -\frac{1}{1-\nu} \frac{1}{\bar{h}^2} \frac{1}{\bar{c}_3} \bar{M}_T, \quad \bar{x} = 0, 1, \\ \bar{w} = 0, \quad \frac{\partial^2 \bar{w}}{\partial \bar{y}^2} = -\frac{1}{1-\nu} \frac{1}{\bar{h}^2} \frac{1}{\bar{c}_3} \bar{M}_T, \quad \bar{y} = 0, \bar{b}. \tag{50}$$

Now the dimensionless form of the thermal resultant moment $\bar{M}_T$ is expanded into a double trigonometric series as

$$\bar{M}_T = \sum_{p=\text{odd}}^{\infty} \sum_{q=\text{odd}}^{\infty} F_{pq}(\tau) \sin \alpha_p \bar{x} \, \sin \beta_q \bar{y}, \tag{51}$$

$$F_{pq}(\tau) = \frac{16}{\pi^2}\frac{1}{pq}\left\{\sum_{j=1}^{\infty} \mathrm{e}^{-q_j^2\tau}\int_1^2 D_{1j}(\zeta)\zeta^{m+n}(\zeta - 1 - \bar{\eta})\mathrm{d}\zeta \right.$$
$$\left. + \cos\bar{\omega}\tau \int_1^2 D_2(\zeta)\zeta^{m+n}(\zeta - 1 - \bar{\eta})\mathrm{d}\zeta + \sin\bar{\omega}\tau \int_1^2 D_3(\zeta)\zeta^{m+n}(\zeta - 1 - \bar{\eta})\mathrm{d}\zeta \right\}, \tag{52}$$

$$\alpha_p = p\pi, \quad \beta_q = \frac{q\pi}{\bar{b}}.$$

The definite integrals in (52) are evaluated numerically.

The solution of the quasi-static deflection therefore may be presented in dimensionless form as

$$\bar{w} = \sum_{p=\text{odd}}^{\infty}\sum_{q=\text{odd}}^{\infty} w_{pq}(\tau)\sin\alpha_p\bar{x}\,\sin\beta_q\bar{y}, \tag{53}$$

$$w_{pq}(\tau) = \frac{1}{1-\nu}\frac{1}{\bar{h}^2}\frac{1}{\bar{c}_3}\frac{F_{pq}(\tau)}{\alpha_p^2 + \beta_q^2}. \tag{54}$$

Suppose that the in-plane displacement components at the reference plane are restrained to be zero at edge lines:

$$u = 0 \quad \text{at } x = 0, a, \quad v = 0 \quad \text{at } y = 0, b; \tag{55}$$

the in-plane displacement components at the reference plane become

$$u(x, y) = 0, \quad v(x, y) = 0. \tag{56}$$

Then the equations of (8) reduce to

$$N_x = -\frac{1}{1-\nu}N_{\mathrm{T}}, \quad N_y = -\frac{1}{1-\nu}N_{\mathrm{T}}, \quad N_{xy} = 0. \tag{57}$$

Dimensionless forms of thermal-stress components may be written as

$$\begin{aligned}
\bar{\sigma}_{xx} &= \frac{\bar{h}^2}{1-\nu^2}\zeta^m(\zeta - 1 - \bar{\eta})\sum_{p=\text{odd}}^{\infty}\sum_{q=\text{odd}}^{\infty} w_{pq}(\tau)(\alpha_p^2 + \nu\beta_q^2)\sin\alpha_p\bar{x}\,\sin\beta_q\bar{y} - \frac{1}{1-\nu}\zeta^{m+n}\bar{T},\\
\bar{\sigma}_{yy} &= \frac{\bar{h}^2}{1-\nu^2}\zeta^m(\zeta - 1 - \bar{\eta})\sum_{p=\text{odd}}^{\infty}\sum_{q=\text{odd}}^{\infty} w_{pq}(\tau)(\beta_q^2 + \nu\alpha_p^2)\sin\alpha_p\bar{x}\,\sin\beta_q\bar{y} - \frac{1}{1-\nu}\zeta^{m+n}\bar{T},\\
\bar{\sigma}_{xy} &= -\frac{\bar{h}^2}{1-\nu^2}\zeta^m(\zeta - 1 - \bar{\eta})\sum_{p=\text{odd}}^{\infty}\sum_{q=\text{odd}}^{\infty} w_{pq}(\tau)\alpha_p\beta_q\cos\alpha_p\bar{x}\,\cos\beta_q\bar{y}.
\end{aligned} \tag{58}$$

3.3 Dynamic thermo-elastic response of inhomogeneous plates

We assume that the mass density is given as a power of the thickness coordinate z' as

$$\rho(z') = \rho_0\left(1 + \frac{z'}{h}\right)^{\gamma}, \tag{59}$$

in which the exponent γ is a parameter representing the inhomogeneity in the mass density.

The following dimensionless quantity is defined as

$$\bar{\mu} = \frac{\mu}{E_0 h^3/\kappa_0^2}. \tag{60}$$

Substitution of (59) in (14) with (60) gives

$$\bar{\mu} = \begin{cases} \dfrac{\rho_0 \kappa_0^2}{E_0 h^2} \dfrac{2^{\gamma+1}-1}{\gamma+1}, & \gamma \neq -1, \\ \dfrac{\rho_0 \kappa_0^2}{E_0 h^2} \ln 2, & \gamma = -1. \end{cases} \tag{61}$$

When the plate is not subjected to a transverse load, the equation of motion governing the dynamic deflection is expressed in dimensionless form as

$$\left(\frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2}\right)^2 \bar{w} + \frac{\bar{\mu}}{\bar{c}_3} \frac{1}{\bar{h}^4} \frac{\partial^2 \bar{w}}{\partial \tau^2} = -\frac{1}{1-\nu} \frac{1}{\bar{h}^2} \frac{1}{\bar{c}_3} \left(\frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2}\right) \bar{M}_{\mathrm{T}}. \tag{62}$$

Solving a homogeneous equation, which is given by elimination of the right-hand side in the equation of motion (62), we obtain the solution of free flexural vibration. The natural angular frequency in flexural vibration Ω_{pq} is given as

$$\Omega_{pq} = \left(\frac{\bar{c}_3 \bar{h}^4}{\bar{\mu}}\right)^{\frac{1}{2}} (\alpha_p^2 + \beta_q^2). \tag{63}$$

Expanding the solution of the dynamic deflection satisfying the equation of motion (62) as a superposition of the flexural vibrations of natural modes, and solving, under the following initial conditions

$$\bar{w} = 0, \quad \frac{\partial \bar{w}}{\partial \tau} = 0 \quad \text{at } \tau = 0, \tag{64}$$

we obtain the dynamic solution for the deflection as

$$\bar{w} = \sum_{p=\mathrm{odd}}^{\infty} \sum_{q=\mathrm{odd}}^{\infty} \Theta_{pq}(\tau) \sin \alpha_p \bar{x} \, \sin \beta_q \bar{y}, \tag{65}$$

in which

$$\begin{aligned} \Theta_{pq}(\tau) = {} & \frac{1}{1-\nu} \frac{1}{\bar{c}_3} \frac{1}{\bar{h}^2} \frac{4\bar{b}}{\pi^2} \frac{\alpha_p^2 + \beta_q^2}{pq} \\ & \times \left[\sum_{j=1}^{\infty} \frac{1}{\Omega_{pq}^2 + q_j^4} \left(\mathrm{e}^{-q_j^2 \tau} - \cos \Omega_{pq} \tau + \frac{q_j^2}{\Omega_{pq}} \sin \Omega_{pq} \tau \right) \int_1^2 D_{1j}(\zeta) \zeta^{m+n} (\zeta - 1 - \bar{\eta}) \mathrm{d}\zeta \right. \\ & + \frac{1}{\Omega_{pq}^2 - \bar{\omega}^2} \left(\cos \bar{\omega} \tau - \cos \Omega_{pq} \tau \right) \int_1^2 D_2(\zeta) \zeta^{m+n} (\zeta - 1 - \bar{\eta}) \mathrm{d}\zeta \\ & \left. + \frac{1}{\Omega_{pq}^2 - \bar{\omega}^2} \left(\sin \bar{\omega} \tau - \frac{\bar{\omega}}{\Omega_{pq}} \sin \Omega_{pq} \tau \right) \int_1^2 D_3(\zeta) \zeta^{m+n} (\zeta - 1 - \bar{\eta}) \mathrm{d}\zeta \right]. \end{aligned} \tag{66}$$

If the in-plane displacement components at the reference plane are restrained to be zero at edge lines, the in-plane displacement components u, v at the reference plane become zero. Then, the thermal-stress components are expressed in dimensionless form as

$$\begin{aligned} \bar{\sigma}_{xx} &= \frac{\bar{h}^2}{1-\nu^2} \zeta^m (\zeta - 1 - \bar{\eta}) \sum_{p=\mathrm{odd}}^{\infty} \sum_{q=\mathrm{odd}}^{\infty} \Theta_{pq}(\tau) (\alpha_p^2 + \nu \beta_q^2) \sin \alpha_p \bar{x} \, \sin \beta_q \bar{y} - \frac{1}{1-\nu} \zeta^{m+n} \bar{T}, \\ \bar{\sigma}_{yy} &= \frac{\bar{h}^2}{1-\nu^2} \zeta^m (\zeta - 1 - \bar{\eta}) \sum_{p=\mathrm{odd}}^{\infty} \sum_{q=\mathrm{odd}}^{\infty} \Theta_{pq}(\tau) (\beta_q^2 + \nu \alpha_p^2) \sin \alpha_p \bar{x} \, \sin \beta_q \bar{y} - \frac{1}{1-\nu} \zeta^{m+n} \bar{T}, \\ \bar{\sigma}_{xy} &= -\frac{\bar{h}^2}{1-\nu^2} \zeta^m (\zeta - 1 - \bar{\eta}) \sum_{p=\mathrm{odd}}^{\infty} \sum_{q=\mathrm{odd}}^{\infty} \Theta_{pq}(\tau) \alpha_p \beta_q \cos \alpha_p \bar{x} \, \cos \beta_q \bar{y}, \end{aligned} \tag{67}$$

in which the normal components of the thermal stress $\bar{\sigma}_{ii} (i = x, y)$ can be expressed as the summation of the components of bending stress, which are expressed as $\bar{\sigma}_{ii-\mathrm{out}} (i = x, y)$ and the component produced by restrained from free thermal expansion, which is expressed as $\bar{\sigma}_{\mathrm{T}}$.

Table 1 Material properties of mild steel

Young's modulus E_0	206.0	[GPa]
Coefficient of linear thermal expansion α_0	11.5×10^{-6}	[1/K]
Poisson's ratio ν	0.28	
Mass density ρ_0	7.85×10^3	[kg/m^3]
Thermal diffusivity κ_0	2.02×10^{-5}	[m^2/s]

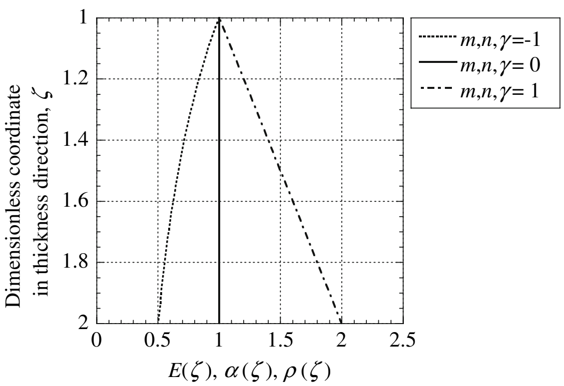

Fig. 3 Variation of material property in the thickness direction

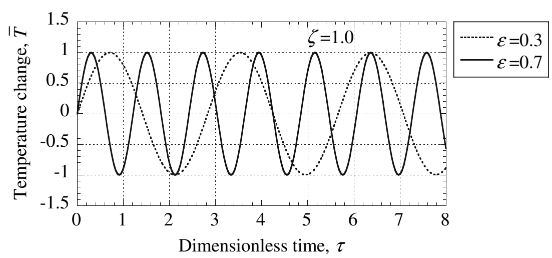

Fig. 4 Difference in time evolution of temperature change at the upper boundary surface $\zeta = 1$ due to frequency in cyclic heating

4 Numerical results and discussion

We perform numerical calculations and examine the effects of material inhomogeneities, angular frequency in cyclic heating and aspect ratio of a rectangular plate on the dynamical thermo-elastic response. The length of the plate a, its thickness h and aspect ratio $\bar{b} = b/a$ are given as

$$a = 5.01\,[\text{m}], \quad h = 5 \times 10^{-3}\,[\text{m}], \quad \bar{b} = \{1, 1.5, 2, 2.5, 3\}. \tag{68}$$

Typical values of the material properties are assumed to be those of mild steel as shown in Table 1. The angular frequency in cyclic heating $\bar{\omega}$ is given as

$$\bar{\omega} = \varepsilon \Omega_{11}^{(\text{h-square})}, \tag{69}$$

in which ε is a parameter; $\Omega_{11}^{(\text{h-square})}$ is a dimensionless fundamental natural angular frequency in flexural vibration for a homogeneous square plate.

We assume that the inhomogeneity parameters in the specific heat and the thermal conductivity k, l are set at zero, and that the parameters in Young's modulus, the coefficient of linear thermal expansion and the mass density are independently given as

$$k = l = 0, \quad m, n, \gamma = \{-1,\ -0.5,\ 0,\ 0.5,\ 1\}. \tag{70}$$

Figure 3 shows the variation of the material property in the thickness direction due to the inhomogeneity parameter. When the inhomogeneity parameter is equal to zero, the material property is constant through the thickness direction, which corresponds to a homogeneous medium. When the parameter is equal to one or minus one, the material property increases linearly with the dimensionless thickness coordinate ζ, or the property varies inversely with the coordinate ζ. To examine the effect of each material inhomogeneity, one of the parameters is varied independently while the others are set to zero.

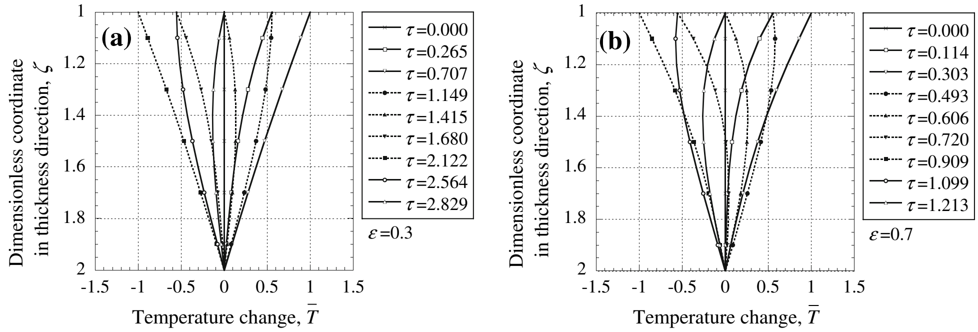

Fig. 5 Difference in variation of temperature change in the thickness direction due to frequency in cyclic heating. (**a**) $\varepsilon = 0.3$, (**b**) $\varepsilon = 0.7$

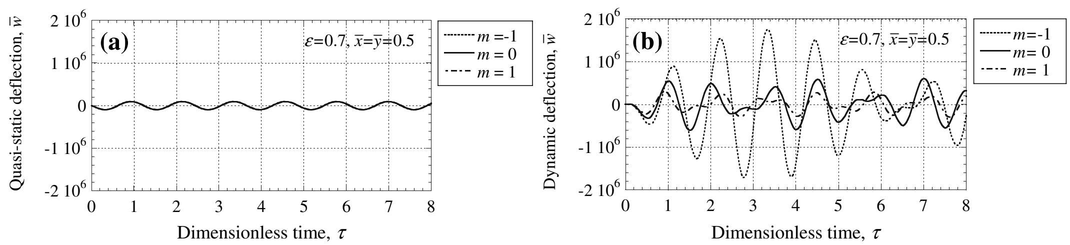

Fig. 6 Difference in time evolution of deflection due to the inhomogeneity in Young's modulus. (**a**) Quasi-static behavior of deflection; (**b**) Dynamic response of deflection

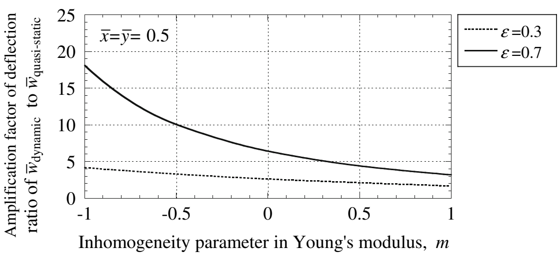

Fig. 7 Effect of inhomogeneity in Young's modulus on the amplification factor of deflection

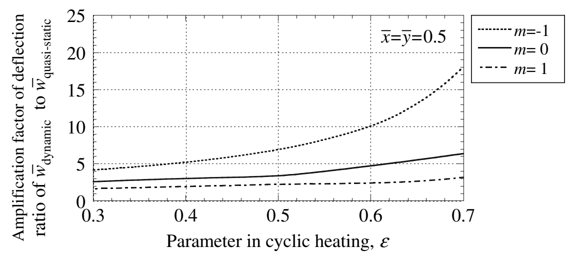

Fig. 8 Effect of frequency in cyclic heating on the amplification factor of deflection

Figures 4 and 5 show the differences in time evolution of the temperature change at the upper boundary surface $\zeta = 1$, and in the variation of temperature change in the thickness direction due to frequency parameters in cyclic heating, $\varepsilon = 0.3$, 0.7. Because the temperatures at the upper and lower boundary surfaces are prescribed, the variation of angular frequency in cyclic heating does not affect the temperature difference between the upper and lower boundary surfaces. The amplitude of temperature change in the plate at $\varepsilon = 0.7$ is slightly less than that at $\varepsilon = 0.3$. This is because the phase lag in the temperature change between the upper boundary surface and the inside of the plate are larger for $\varepsilon = 0.7$ than for $\varepsilon = 0.3$.

Now we examine the effect of inhomogeneity in Young's modulus on the thermo-elastic response of the plate. Figure 6 shows the difference in time evolutions of a quasi-static deflection and of a dynamic deflection for $\varepsilon = 0.7$ due to the inhomogeneity parameter in Young's modulus m. The time evolution of the quasi-static deflection hardly changes due to variation of the parameter m. The smaller the parameter m, the larger the maximum amplitude in the dynamic deflection will be. A beat-wise oscillation in the dynamic deflection is seen for $m = -1$. Figure 7 indicates the effect of the inhomogeneity in Young's modulus on the amplification factor of deflection for frequency parameters in cyclic heating equal to $\varepsilon = 0.3$ and $\varepsilon = 0.7$. Here the amplification factor of deflection is defined by

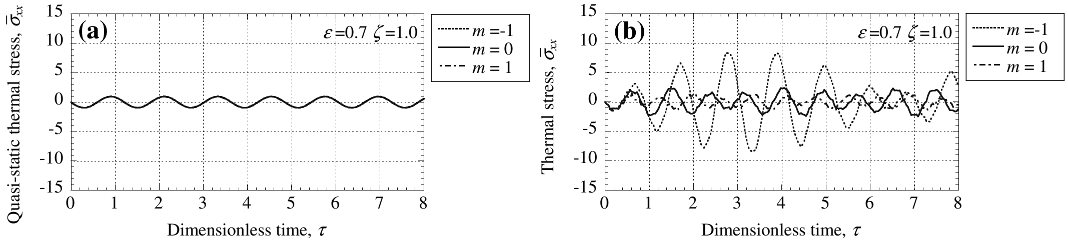

Fig. 9 Difference in time evolution of thermal-stress component $\bar{\sigma}_{xx}$ at the upper surface $\zeta = 1$ due to the inhomogeneity in Young's modulus. (**a**) Quasi-static behavior of thermal-stress component $\bar{\sigma}_{xx}$; (**b**) Dynamic response of thermal-stress component $\bar{\sigma}_{xx}$

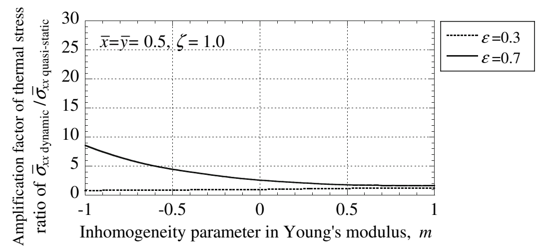

Fig. 10 Effect of inhomogeneity in Young's modulus on the amplification factor of thermal-stress $\bar{\sigma}_{xx}$ at the upper surface $\zeta = 1$

the ratio of the maximum amplitude of dynamic deflection to that of quasi-static deflection. As the inhomogeneity parameter in Young's modulus m decreases, the amplification factor of deflection increases. The variation of the amplification factor of deflection with the parameter m for $\varepsilon = 0.7$ is larger than that for $\varepsilon = 0.3$. Figure 8 indicates the effect of frequency in cyclic heating on the amplification factor of deflection. The amplification factor of deflection increases with the parameter in cyclic heating ε. The amplification factor of deflection increases with the frequency parameter in cyclic heating ε. The variation of the amplification factor of deflection for $m = -1$ is the largest among those of the three parameters. The dimensionless form of the fundamental natural angular frequency of the inhomogeneous square plate Ω_{11} is expressed as

$$\Omega_{11} = 2\pi^2\sqrt{\frac{\bar{c}_3\bar{h}^4}{\bar{\mu}}}. \tag{71}$$

As the inhomogeneity parameter in Young's modulus m decreases, the flexural rigidity $\bar{c}_3$ decreases, and thus the dimensionless fundamental natural angular frequency Ω_{11} decreases. Then the dimensionless fundamental natural angular frequency Ω_{11} becomes closer to the dimensionless angular frequency in cyclic heating $\bar{\omega}$. Consequently, this induces an increase in the amplitude of the flexural vibration, and a beat-wise time evolution of the dynamic deflection occurs. Figure 9 shows the difference in time evolution of the quasi-static behavior and the dynamic response of the thermal-stress component $\bar{\sigma}_{xx}$ at the upper surface $\zeta = 1$ due to the inhomogeneity parameter in Young's modulus m. The parameter m hardly has an effect on the quasi-static behavior of the thermal-stress component $\bar{\sigma}_{xx}$ at the upper surface. Meanwhile the amplitude of the dynamic response of the thermal-stress component $\bar{\sigma}_{xx}$ at the upper surface increases as the inhomogeneity parameter in Young's modulus m decreases.

Figure 10 indicates the effect of the inhomogeneity in Young's modulus on the amplification factor of the thermal-stress component $\bar{\sigma}_{xx}$ at the upper surface $\zeta = 1$ for frequency parameters in cyclic heating equal to $\varepsilon = 0.3$ and $\varepsilon = 0.7$. Here the amplification factor of thermal-stress component $\bar{\sigma}_{xx}$ at the upper surface $\zeta = 1$ is defined by the ratio of maximum amplitude of the dynamic thermal-stress component $\bar{\sigma}_{xx}$ at the upper surface $\zeta = 1$ to that of the quasi-static thermal-stress component $\bar{\sigma}_{xx}$ at the upper surface. The inhomogeneity parameter in Young's modulus m hardly affects the amplification factor of the thermal-stress component $\bar{\sigma}_{xx}$ at the upper surface for $\varepsilon = 0.3$. The amplification factor of the thermal-stress component $\bar{\sigma}_{xx}$ at the upper surface for $\varepsilon = 0.7$ increases as the parameter m decreases. Because the angular frequency in cyclic heating $\bar{\omega}$ for $\varepsilon = 0.7$ is close to the dimensionless

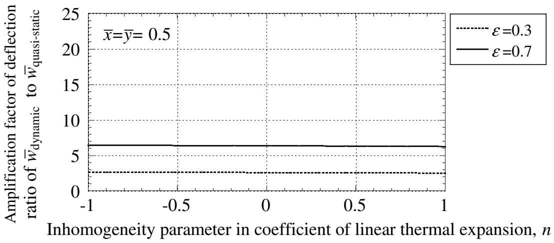

Fig. 11 Effect of inhomogeneity in the coefficient of linear thermal expansion on the amplification factor of deflection

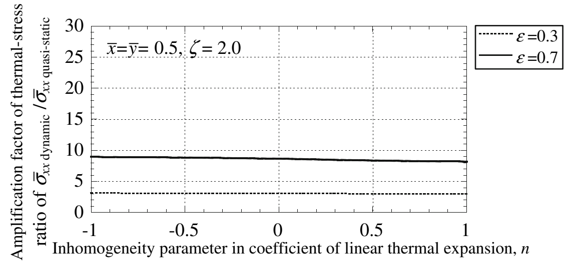

Fig. 12 Effect of inhomogeneity in the coefficient of linear thermal expansion on the amplification factor of thermal stress $\bar{\sigma}_{xx}$ at the lower surface $\zeta = 2$

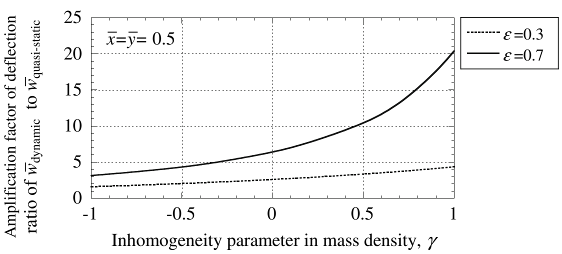

Fig. 13 Effect of inhomogeneity in mass density on the amplification factor of deflection

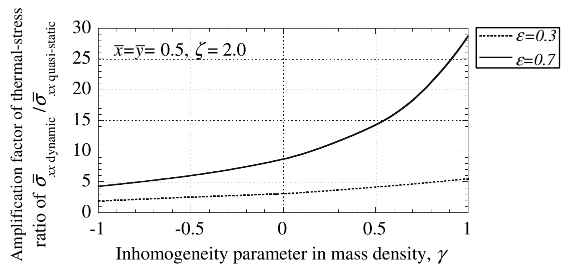

Fig. 14 Effect of inhomogeneity in mass density on the amplification factor of thermal stress $\bar{\sigma}_{xx}$ at the lower surface $\zeta = 2$

fundamental natural angular frequency Ω_{11}, the amplitude of the bending-stress component $\bar{\sigma}_{xx\text{-out}}$ at the upper surface increases with the amplitude of the dynamic deflection.

Next, we examine an effect of the inhomogeneity in the coefficient of linear thermal expansion on the thermo-elastic response of the plate. Figure 11 indicates the effect of the inhomogeneity in the coefficient of linear thermal expansion on the amplification factor of deflection for frequency parameters in cyclic heating $\varepsilon = 0.3$ and $\varepsilon = 0.7$. The amplification factor of deflection hardly changes with the parameter n. The amplification factor of deflection for $\varepsilon = 0.7$ is larger than that for $\varepsilon = 0.3$. Figure 12 indicates the effect of the inhomogeneity in coefficient of linear thermal expansion on the amplification factor of the thermal-stress component $\bar{\sigma}_{xx}$ at the lower surface $\zeta = 2$ for frequency parameters in cyclic heating $\varepsilon = 0.3$ and $\varepsilon = 0.7$. The amplification factor of the thermal-stress component $\bar{\sigma}_{xx}$ at the lower surface hardly changes with the parameter n. The amplification factor of the thermal-stress component $\bar{\sigma}_{xx}$ at the lower surface for $\varepsilon = 0.7$ is larger than that for $\varepsilon = 0.3$.

Next, we examine the effect of inhomogeneity in mass density on the thermo-elastic response of the plate. Figure 13 indicates the effect of the inhomogeneity in mass density on the amplification factor of deflection for $\varepsilon = 0.3$ and $\varepsilon = 0.7$. The amplification factor of deflection increases with the inhomogeneity parameter in mass density γ. The variation of the amplification factor of deflection for $\varepsilon = 0.7$ is larger than that for $\varepsilon = 0.3$. Figure 14 indicates the effect of the inhomogeneity in mass density on the amplification factor of thermal-stress component $\bar{\sigma}_{xx}$ at the lower surface $\zeta = 2$ for $\varepsilon = 0.3$ and $\varepsilon = 0.7$. The amplification factor of thermal-stress component $\bar{\sigma}_{xx}$ at the lower surface $\zeta = 2$ increases with the parameter γ. The variation of amplification factor of the thermal-stress component $\bar{\sigma}_{xx}$ with the parameter γ for $\varepsilon = 0.7$ is larger than that for $\varepsilon = 0.3$.

Figures 15 and 16 show the cubic diagrams and the cross-section diagrams of the dynamic deflection of rectangular plates at $\bar{x} = 0.5$ with aspect ratios $\bar{b} = 1$ and $\bar{b} = 3$, respectively.

From Figure 16a the deflection curve shows the fundamental natural mode of flexural vibration in a square plate of aspect ratio $\bar{b} = 1$. In the rectangular plate of aspect ratio $\bar{b} = 3$, in addition to the fundamental natural mode of the deflection curve, a higher-order mode of deflection curve in the longitudinal direction $\bar{y}$ can be seen in Fig. 16b.

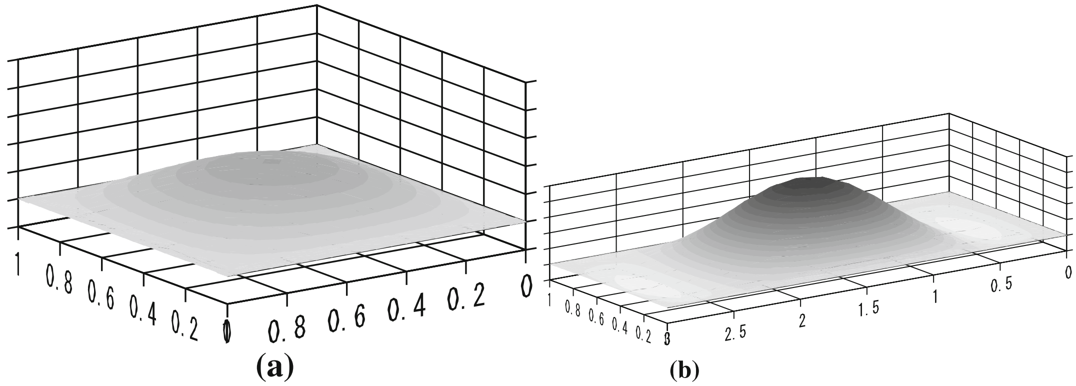

Fig. 15 Difference in the cubic diagrams of dynamic deflection due to aspect ratio of plate. (**a**) Square plate of $\bar{b} = 1$; (**b**) Rectangular plate of $\bar{b} = 3$

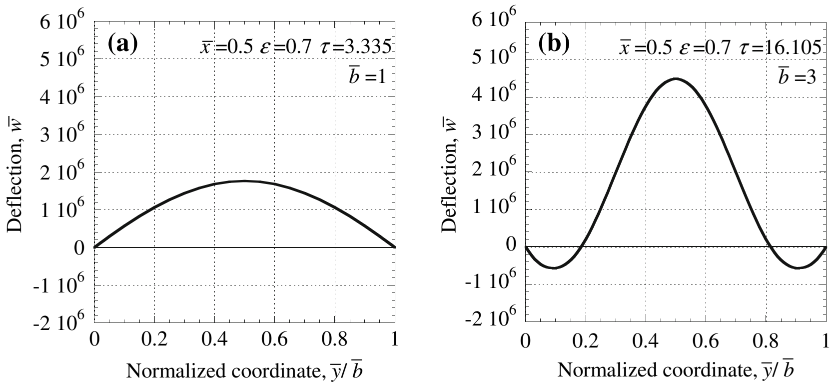

Fig. 16 Difference in distribution of dynamic deflection due to aspect ratio of plate. (**a**) Square plate of $\bar{b} = 1$. (**b**) Rectangular plate of $\bar{b} = 3$

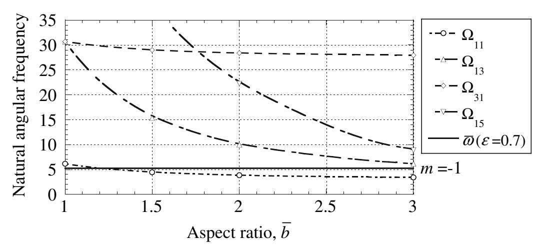

Fig. 17 Variation of natural angular frequencies due to aspect ratio of plate

Figure 17 shows the effect of the aspect ratio of the plate on the natural frequencies for the inhomogeneity parameter in Young's modulus $m = -1$. The natural angular frequencies Ω_{11} and Ω_{13} in a plate of the aspect ratio $\bar{b} = 1$ are larger than the angular frequency in cyclic heating $\bar{\omega}$ for $\varepsilon = 0.7$. The natural angular frequencies of higher-order modes in the longitudinal direction $\bar{y}$ decrease considerably with an increase in aspect ratio $\bar{b}$. The natural angular frequency Ω_{13} in the rectangular plate of aspect ratio $\bar{b} = 3$ is closer to the angular frequency in cyclic heating $\bar{\omega}$ for $\varepsilon = 0.7$ than the fundamental natural frequency Ω_{11}. Consequently, this forms the higher-order mode of the flexural vibration in the longitudinal direction.

5 Concluding remarks

The equations of motion governing thermally induced vibration of plates with inhomogeneous material properties through the thickness direction have been presented. Subsequently, equations of motion of thermally induced flexural vibration for inhomogeneous rectangular plates in which material properties are given as a power of the thickness coordinate have been derived from the above-mentioned fundamental equations. An exact solution of the one-dimensional temperature change has been obtained for an inhomogeneous plate in which one surface is exposed to a sinusoidally varying temperature, and the other is kept at zero temperature change. The associated quasi-static and dynamic solutions pertinent to deflection and thermal stresses in the inhomogeneous rectangular plate have been derived for a simply supported edge condition. Numerical calculations have been performed, and the effects of material inhomogeneity such as Young's modulus, coefficient of linear thermal expansion, and mass density, angular frequency in cyclic heating, and aspect ratio on thermo-elastic response of the rectangular plate have been examined.

Numerical results are summarized as follows:
The larger the angular frequency in the cyclic heating, the smaller the amplitude of the temperature change inside the plate will be. Because temperatures at the upper and lower boundary surfaces are prescribed, the variation of angular frequency in cyclic heating does not affect the temperature difference between upper and lower boundary surfaces. The variation of the inhomogeneity parameter of Young's modulus m hardly affects the amplification of the quasi-static deflection. As the inhomogeneity parameter of Young's modulus m decreases, the maximum amplitude of the dynamic deflection increases, and the shape of the time evolution of the dynamic deflection changes. Consequently, the amplification factor of the deflection increases as the inhomogeneity parameter of Young's modulus m decreases. The inertia effect in the deflection is most notable for the parameter in cyclic heating $\varepsilon = 0.7$. As the maximum amplitude of the dynamic deflection increases, the amplitude of the subsequent dynamic thermal stresses increases. The amplification factor of the dynamic deflection increases with the angular frequency in cyclic heating, because the angular frequency in cyclic heating becomes close to the fundamental natural angular frequency of the plate. The variation of the inhomogeneity parameter in the coefficient of linear thermal expansion n hardly affects the amplitude and the time evolution of the thermal resultant moment. The inhomogeneity parameter in the coefficient of linear thermal expansion n hardly affects the amplification factors of the deflection and the thermal stress. The amplification factors of the deflection and the thermal stress increase with the inhomogeneity parameter in mass density γ. The inertia effect in the deflection and the subsequent thermal stress is clearly visible for a parameter in cyclic heating equal to $\varepsilon = 0.7$. As the aspect ratio of the plate $\bar{b}$ increases, higher-order mode curves of the deflection and the thermal stress in the longitudinal direction are seen. As the aspect ratio of the plate $\bar{b}$ increases, the natural angular frequencies of higher-order modes in the longitudinal direction decrease considerably, coming close to the angular frequency in cyclic heating.

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